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On the Cauchy problem for the Schrödinger equation with superoscillatory initial data

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Abstract

Superoscillatory functions were introduced in Aharonov and Vaidman (1990) [5], and recently studied in detail in Aharonov et al. (2011) [2], Berry (1994) [7] and Berry and Popescu (2006) [9]. In this paper we study the time evolution of a superoscillating function, by taking it as initial value for the Cauchy problem for the Schrödinger equation. By using convolution operators on spaces of entire functions with suitable growth conditions, we prove the surprising fact that the superoscillatory phenomenon persists for all values of t.

Résumé

Les fonctions superoscillation ont été introduites dans Aharonov et Vaidman (1990) [5], et ont été récemment étudiées en détail dans Aharonov et al. (2011) [2], Berry (1994) [7] et Berry et Popescu (2006) [9]. Dans cet article, on étudie l'évolution temporelle d'une fonction superoscillante qui est prise pour valeur initiale d'un le problème de Cauchy pour l'équation de Schrödinger. En utilisant des opérateurs de convolution sur les espaces de fonctions entières à croissance rapide, on démontre le fait surprenant que le phénomène de superoscillation persiste pour toutes les valeurs de t.

Keywords: Superoscillatory functions; Entire functions with growth conditions; Schrödinger equation; Cauchy problem

1. Introduction

In [5], Aharonov and Vaidman discovered, as a consequence of their far reaching theory of weak measurement, the existence of what they called *superoscillating* functions, namely functions which exhibit oscillatory behavior in what might appear at first sight to be a violation of the fundamental theorem of Fourier Analysis. Partly because of the applications of such functions, and partly in view of the growing importance of the notion of weak measurement, there have been several papers devoted to the study of the physics of superoscillations and recently a few papers have appeared that discuss the mathematical significance of such superoscillating functions, e.g. [2,3].

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An important question concerning superoscillating functions deals with whether the superoscillatory behavior persists when superoscillating functions are evolved in time as solutions of the Schrödinger equation. That this should be the case has been experimentally established for example in [1] and in [4]. Given that the magnitude of the superoscillating region is created at the expense of exponentially larger regions, one might think that the superoscillations would not last long. However, [9] showed that this is not the case: the superoscillations last for a long time. Furthermore, [9] showed that the superoscillations can be used in a practical sense for superresolution of very fine details, even without the traditional use of evanescent waves. We refer the reader to [7–9] for a more thorough discussion of the physical background of this problem (in which a different superoscillating function is studied with different methods).

The purpose of this paper is to investigate the superoscillating behavior of the solution to the Schrödinger equation whose initial value is the superoscillating function

$$F_n(x) := \left(\cos\left(\frac{x}{nL}\right) + ia\sin\left(\frac{x}{nL}\right)\right)^n,$$

and to confirm that the superoscillatory phenomenon persists for all values of t.

Without loss of generality, in the sequel we will set $L = 1$, and we will study the Cauchy problem

$$i\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t), \qquad \psi(x,0) = F_n(x),$$

where

$$H\psi(x,t) := -\frac{\partial^2\psi(x,t)}{\partial x^2}.$$

The following result is easily established:

Theorem 1.1. *The time evolution of the spatial superoscillating function $F_n(x)$, is given by*

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a)e^{i(1-2k/n)x}e^{-it(1-2k/n)^2}.$$

Proof. In order to solve the Schrödinger equation with $F_n(x)$ as initial condition we will work in the space of the tempered distributions $\mathcal{S}'(\mathbb{R})$ and use a standard Fourier transform argument. Let us consider the Fourier and anti-Fourier transforms:

$$\mathcal{F}\big[\psi(x,t)\big] := \int_{\mathbb{R}} \psi(x,t)e^{-ipx}\,dx, \qquad \mathcal{F}^{-1}\big[g(p,t)\big] := \frac{1}{2\pi}\int_{\mathbb{R}} g(p,t)e^{ipx}\,dp$$

and let us set, for simplicity, $\mathcal{F}[\psi(x,t)] = \hat{\psi}(p,t)$. Taking the Fourier transform of the Schrödinger equation we get

$$i\frac{d\hat{\psi}(p,t)}{dt} = p^2\hat{\psi}(p,t)$$

and integrating we obtain

$$\hat{\psi}(p,t) = C(p)e^{-ip^2t},$$

where the arbitrary function $C(p)$ can be determined by the initial condition

$$C(p) = \hat{\psi}(p,0) = \int_{\mathbb{R}} \left[\sum_{k=0}^{n} C_k(n,a)e^{i(1-2k/n)x}\right]e^{-ipx}\,dx$$
$$= \sum_{k=0}^{n} C_k(n,a)\int_{\mathbb{R}} e^{i(1-2k/n)x}e^{-ipx}\,dx.$$

Here we use the fact that $\mathcal{F}(e^{imx}) = 2\pi\delta(x-m)$, which has to be interpreted in $\mathcal{S}'(\mathbb{R})$. Thus we have

$$C(p) = \sum_{k=0}^{n} C_k(n,a)\delta\big(p-(1-2k/n)\big)$$

and

$$\hat{\psi}(p,t) = \sum_{k=0}^{n} C_k(n,a)\delta\big(p-(1-2k/n)\big)e^{-ip^2t}.$$

Taking now $\mathcal{F}^{-1}$ we have

$$\begin{aligned}\psi(x,t) &= \int_{\mathbb{R}} \left[\sum_{k=0}^{n} C_k(n,a)\delta\big(p-(1-2k/n)\big)e^{-ip^2t}\right]e^{ipx}\,dp \\ &= \sum_{k=0}^{n} C_k(n,a)\int_{\mathbb{R}} \big[\delta\big(p-(1-2k/n)\big)e^{-ip^2t}\big]e^{ipx}\,dp \\ &= \sum_{k=0}^{n} C_k(n,a)\int_{\mathbb{R}} \delta\big(p-(1-2k/n)\big)e^{ipx}e^{-ip^2t}\,dp \\ &= \sum_{k=0}^{n} C_k(n,a)e^{i(1-2k/n)x}e^{-it(1-2k/n)^2}. \qquad \Box\end{aligned}$$

In this paper we will prove that the function $\psi_n(x,t)$ exhibits, for all values of t, the same superoscillatory behavior shown by $F_n(x)$, and we calculate an approximate value for the error $|\psi_n(x,t)-e^{iax-ia^2t}|$ for n large, where e^{iax-ia^2t} is the limit function of $\psi_n(x,t)$.

The plan of the paper is the following. In Section 2 we recall some properties of the superoscillating function F_n and we prove some estimates for its first and second derivatives which will be useful in the sequel. Section 3 contains our main results. Specifically we will write the function ψ_n in terms of power series expansion of differential operators acting on F_n (this is in fact the formal power series expansion of the evolution operator e^{itH}); we will then study the operator e^{itH} with functional analysis techniques, in order to compute

$$\lim_{n\to+\infty} \psi_n(x,t) = e^{iax-ia^2t},$$

and we will then obtain estimates of $|\psi_n(x,t)-e^{ia(x-at)}|$.

2. The superoscillating functions F_n and their derivatives

We start this section by recalling the identity

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \sum_{k=0}^{n} C_k(n,a)e^{i(1-2k/n)x} \tag{1}$$

for $a \in \mathbb{R}$, $x \in \mathbb{R}$ where

$$C_k(n,a) = \binom{n}{k}\left(\frac{1+a}{2}\right)^{n-k}\left(\frac{1-a}{2}\right)^{k},$$

and we set

$$F(x,a) = e^{iax}. \tag{2}$$

For physical reasons, the interesting case will be for $a>1$. In [2] we have proved the following result.

Theorem 2.1. *Let $F_n(x,a)$ and $F(x,a)$ be defined in* (1) *and* (2)*, respectively. We have*:

(1) *Let $n \in \mathbb{N}$ be fixed and $a>1$. Then*

$$\sup_{x\in\mathbb{R}} \big|F_n(x,a)\big| = a^n.$$

(2) *The sequence $F_n(x,a)$ converges to $F(x,a)$ for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$, but it does not converge uniformly.*
(3) *Let $M > 0$ be a fixed real number. Then for every x such that $|x| \leqslant M$ the sequence $F_n(x,a)$ converges uniformly to $F(x,a)$.*
(4) *The module of the difference $F_n(x,a) - F(x,a)$ can be written as follows*:

$$\left|F_n(x,a) - F(x,a)\right|^2 = 1 + \left(\cos^2\left(\frac{x}{n}\right) + a^2 \sin^2\left(\frac{x}{n}\right)\right)^n - 2\left(\cos^2\left(\frac{x}{n}\right) + a^2 \sin^2\left(\frac{x}{n}\right)\right)^{n/2} \cos\left[n \arctan\left(a \tan\left(\frac{x}{n}\right)\right) - ax\right].$$

(5) *The approximated error ε for $|F_n(x) - e^{iax}|$ is given by*

$$\varepsilon \sim \frac{|x|}{n}\sqrt{\frac{3}{2}\left(a^2 - 1\right)},$$

for x in a compact set.

We now offer a simple estimate for the first and second derivatives of the functions F_n.

Proposition 2.2. *Let F_n be the function defined in (1). Then for x in a compact set in $\mathbb{R}$ and for n large we have*:

$$\left|F_n'(x)\right| \sim a, \qquad \left|F_n''(x)\right| \sim a^2.$$

Proof. Compute the derivatives of the function $F_n(x) = g_n^n(x)$, where

$$g_n(x) = \cos\left(\frac{x}{n}\right) + ia \sin\left(\frac{x}{n}\right).$$

It is immediate to see that

$$F_n'(x) = n g_n^{n-1} g_n'(x) = n \frac{g_n'(x)}{g_n(x)} F_n(x),$$

and

$$F_n''(x) = \left(-\frac{1}{n} + n(n-1)\left(\frac{g_n'(x)}{g_n(x)}\right)^2\right) F_n(x).$$

We are now interested in the asymptotic behavior of F_n' and F_n'' when n goes to infinity and x is limited to a compact subset of $\mathbb{R}$. We know from [2] that $|F_n(x)|$ converges to 1 on every compact subset of $\mathbb{R}$, so it is enough to compute the asymptotic behavior of g_n'/g_n. In an analogous way we obtain:

$$\frac{g_n'(x)}{g_n(x)} = \frac{1}{n} \frac{-\sin(\frac{x}{n}) + ia\cos(\frac{x}{n})}{\cos(\frac{x}{n}) + ia \sin(\frac{x}{n})} \sim \frac{1}{n} ia$$

for n large and x in a compact set. We then immediately obtain the statement. □

3. The main result

We begin this section by giving an equivalent representation of the time evolution ψ_n in terms of the derivatives of the functions F_n.

Theorem 3.1. *The function*

$$\psi_n(x,t) = \sum_{k=0}^{n} C_k(n,a) e^{ix(1-2k/n)} e^{-it(1-2k/n)^2} \tag{3}$$

can be written as

$$\psi_n(x,t)=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\frac{d^{2m}}{dx^{2m}}F_n(x)$$

for every $x\in\mathbb{R}$ and $t\in\mathbb{R}$.

Proof. We consider the expansion

$$e^{-it(1-2k/n)^2}=\sum_{m=0}^{\infty}\frac{[-it(1-2k/n)^2]^m}{m!}$$

so we get

$$\psi_n(x,t)=\sum_{m=0}^{\infty}\frac{(-it)^m}{m!}\sum_{k=0}^{n}C_k(n,a)(1-2k/n)^{2m}e^{ix(1-2k/n)}$$

which can be written as

$$\begin{aligned}\psi_n(x,t)&=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\sum_{k=0}^{n}C_k(n,a)\frac{d^{2m}}{dx^{2m}}e^{ix(1-2k/n)}\\&=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\frac{d^{2m}}{dx^{2m}}\sum_{k=0}^{n}C_k(n,a)e^{ix(1-2k/n)}\\&=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\frac{d^{2m}}{dx^{2m}}F_n(x).\qquad\square\end{aligned}$$

Remark 3.2. If one were to proceed formally, one could show that

$$\lim_{n\to\infty}\psi_n(x,t)=e^{iax-ia^2t}.$$

In fact, let us take the limit and recall that

$$F_n(x)\to e^{iax},$$

so we obtain

$$\begin{aligned}\psi(x,t)&=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\frac{d^{2m}}{dx^{2m}}e^{iax}\\&=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}(ia)^{2m}e^{iax}\\&=\sum_{m=0}^{\infty}\frac{(-ia^2t)^m}{m!}e^{iax}\\&=e^{iax-ia^2t}.\end{aligned}$$

We will now show that the formal computation in Remark 3.2 can be justified in a rigorous way.

As it is well known, see [11], an operator of the form

$$\sum_{m=0}^{\infty}b_m(z)\frac{d^m}{dz^m}$$

is an infinite-order differential operator which acts continuously on holomorphic functions in $\mathbb{C}$ if and only if, for every compact set $\mathcal{K} \subset \mathbb{C}$,

$$\lim_{k\to\infty} \sqrt[k]{\sup_{z\in\mathcal{K}} \left|b_k(z)\right| k!} = 0.$$

As a consequence, we see that the operator

$$U(t) := \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dz^{2m}}$$

is not continuous on the space of entire functions even for small values of t. It is however possible to find a suitable space of entire function on which $U(t)$ acts continuously. To this purpose, consider the operator

$$\sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^m}{dz^m}$$

acting on holomorphic functions, and denote by g the function which is its symbol. Then $g(\zeta) = \sum_{m=0}^{\infty} \frac{a_m}{m!} \zeta^m$, where $a_m = (it)^m$. It is immediate to show that, unless $t = 0$, this symbol does not define an infinite-order differential operator (in the sense of it being a local operator). However, there are many choices of a_m for which $g(\zeta)$ can be thought of as the symbol of a suitable convolution operator (infinite-order differential operators are examples of convolution operators supported by the origin). For instance, if $a_m \equiv 1$ then $g(\zeta) = e^{\zeta}$ and g is the symbol of the translation of the unit translation operator which is nothing but the convolution with the Dirac delta centered at $z = -1$. Generally speaking if X an Analytically Uniform space (AU-space) as defined by Ehrenpreis in [10], then convolutors on X can be defined in a standard way as follows: let $\mathcal{F}X'$ be the space of the Fourier (or Fourier–Borel) transform of the elements of the dual of X. The definition of AU-space implies that $\mathcal{F}X'$ is a space of entire functions which satisfy suitable growth conditions. Let F be an entire function which, by multiplication, defines a continuous map from $\mathcal{F}X'$ to itself. Then a convolutor on X is the continuous operator on X defined as the adjoint of the map that associates to $f \in X'$ the element

$$\mathcal{F}^{-1}\big(F\big(\mathcal{F}(f)\big)\big).$$

Within this framework, if X is the space of entire functions on $\mathbb{C}$ then X' is the space of analytic functionals and $\mathcal{F}X'$ is (topologically isomorphic to) the space $\mathrm{Exp}(\mathbb{C})$ of entire functions with exponential growth.

As we pointed out before, the function $g(\zeta) = e^{\zeta}$ is clearly a multiplier on $\mathrm{Exp}(\mathbb{C})$ and the convolutor that it defines is the translation as indicated above. If we now consider the symbol $h(\zeta) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \zeta^m$ it is easy to see that such operator defines, for any value of t, a convolutor on $\mathrm{Exp}(\mathbb{C})$. This is easily seen because $h(\zeta) = g(it\zeta)$, and therefore the growth of h is controlled in the same way in which we control the growth of g. The operator we are now interested in, however, is

$$\sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}}$$

whose symbol is $h(\zeta^2)$. It is obvious to see that $h(\zeta^2)$ does not define a multiplication operator on $\mathrm{Exp}(\mathbb{C})$ because it grows at infinity too fast. It is therefore natural to ask whether it is possible to think of an appropriate space for which h would be a multiplier and therefore an appropriate space for which h would induce a convolution operator. The answer to this question is given by the following result:

Theorem 3.3. *For any value of t, the operator $U(t)$ acts continuously on the space*

$$A_{2,0} := \left\{ f \in \mathcal{O}(\mathbb{C}) \mid \forall \varepsilon > 0\ \exists a_\varepsilon \mid \left|f(z)\right| \leqslant a_\varepsilon e^{\varepsilon |z|^2} \right\}$$

of entire functions of order less or equal 2 *and of minimal type.*

Proof. We need to identify a space on which the operator

$$U(t) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m}}{dx^{2m}}$$

would act continuously. Let us therefore consider the space

$$A_2 := \{ f \in \mathcal{O}(\mathbb{C}):\ \exists a, b > 0:\ |f(\zeta)| \leqslant a e^{b|\zeta|^2} \}.$$

It is immediate to verify that $h(\zeta) = e^{\zeta^2}$ is a continuous multiplier on A_2, regardless of the magnitude of t. In [6,12], it is shown that any multiplier on A_2 defines a convolutor on the space

$$A_{2,0} := \{ f \in \mathcal{O}(\mathbb{C}):\ \forall \varepsilon > 0\ \exists a_\varepsilon:\ |f(z)| \leqslant a_\varepsilon e^{\varepsilon |z|^2} \}.$$

We therefore have that for every function $f \in A_{2,0}$, and every $t \in \mathbb{R}$, the function

$$U(t)[f] = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \frac{d^{2m} f}{dz^{2m}}$$

is a well defined function in $A_{2,0}$ and the operator $U(t)$ acts continuously on $A_{2,0}$. □

As a consequence of the previous theorem we can show the main result of this paper, namely that the superoscillatory phenomenon persists for all values of the time t:

Corollary 3.4. *For $a > 1$, and for every $x, t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} \psi_n(x,t) = e^{iax - ia^2 t}.$$

Proof. The functions F_n extend to entire functions of order less than or equal 1 and finite type (i.e. of exponential type), and this space is clearly contained in $A_{2,0}$. □

3.1. An estimate of the error for small times

Proposition 3.5. *For n large the approximated error of $|\psi_n(x,t) - e^{i(ax - a^2 t)}|$ is*

$$\varepsilon(x,t,a,n) = \varepsilon_1(x,t,a,n) + \varepsilon_2(x,t,a,n),$$

where

$$\varepsilon_1(x,t,a,n) = \frac{|x - at|}{n} \sqrt{\frac{3}{2}(a^2 - 1)},$$

and

$$\varepsilon_1(x,t,a,n) = |t|(a^3 + a).$$

Proof. Let us observe that

$$|F_n(x - at) - e^{iax - ia^2 t}| \to 0$$

uniformly on the compact sets of $\mathbb{R}^2$. We now consider the inequality

$$|\psi_n(x,t) - e^{iax - ia^2 t}| \leqslant |\psi_n(x,t) - F_n(x - at)| + |F_n(x - at) - e^{iax - ia^2 t}|.$$

The error of $|F_n(x - at) - e^{iax - ia^2 t}|$ has been estimated in the paper [2], and it is

$$\varepsilon_1(x - at, a, n) = \frac{|x - at|}{n} \sqrt{\frac{3}{2}(a^2 - 1)}.$$

Let us estimate the error associated to the term $|\psi_n(x,t) - F_n(x - at)|$. Consider

$$\psi_n(x,t) - F_n(x - at) = \sum_{k=0}^{n} C_k(n,a) e^{ix(1-2k/n)} e^{-it(1-2k/n)^2} - \sum_{k=0}^{n} C_k(n,a) e^{i(x-at)(1-2k/n)}.$$

It can be written as

$$\psi_n(x,t) - F_n(x-at) = \sum_{k=0}^{n} C_k(n,a) e^{ix(1-2k/n)} \left[e^{-it(1-2k/n)^2} - e^{-iat(1-2k/n)} \right].$$

As a first approximation, as $t \to 0$, we have

$$e^{-it(1-2k/n)^2} - e^{-iat(1-2k/n)} = -it(1-2k/n)^2 + iat(1-2k/n) + o(t)$$

and, as a consequence,

$$\begin{aligned} \psi_n(x,t) - F_n(x-at) &\sim t \sum_{k=0}^{n} C_k(n,a) e^{ix(1-2k/n)} \left[-i(1-2k/n)^2 + ia(1-2k/n) \right] \\ &= t \left[a F_n''(x) - F_n'(x) \right]. \end{aligned}$$

Thus we have

$$\left| \psi_n(x,t) - F_n(x-at) \right| \sim |t| \left| a F_n''(x) - F_n'(x) \right|,$$

and

$$\varepsilon_2(x,t,a,n) := |t| \left| a F_n''(x) - F_n'(x) \right|.$$

Summarizing, the total error is

$$\begin{aligned} \varepsilon(x,t,a,n) &= \varepsilon_1(x,t,a,n) + \varepsilon_2(x,t,a,n) \\ &= \frac{|x-at|}{n} \sqrt{\frac{3}{2}(a^2-1)} + |t| \left| a F_n''(x) - F_n'(x) \right|, \end{aligned}$$

and by Proposition 2.2 we get the statement. □

Finally, let us consider the difference

$$\psi_n(x,t) - e^{i(ax-a^2t)}$$

written as the sum of two contributions as

$$\psi_n(x,t) - e^{i(ax-a^2t)} = Z_n(x,t) + W_n(x,t),$$

where

$$\begin{aligned} Z_n(x,t) &:= \psi_n(x,t) - F_n(x-at), \\ W_n(x,t) &:= F_n(x-at) - e^{i(ax-a^2t)}. \end{aligned}$$

The second contribution can be estimated precisely using Theorem 2

$$\begin{aligned} \left| W_n(x,t) \right|^2 = 1 &+ \left(\cos^2\left(\frac{x-at}{n} \right) + a^2 \sin^2\left(\frac{x-at}{n} \right) \right)^n \\ &- 2 \left(\cos^2\left(\frac{x-at}{n} \right) + a^2 \sin^2\left(\frac{x-at}{n} \right) \right)^{n/2} \cos\left[n \arctan\left(a \tan\left(\frac{x-at}{n} \right) \right) - a(x-at) \right]. \end{aligned}$$

So

$$\left| W_n(x,t) \right|^2 \to 0, \quad \text{for all } x \in [-K,K], \text{ and } t \in [0,T] \text{ uniformly}.$$

For the term $Z_n(x,t)$ it seems to be more delicate to find an analogous estimate, but we can rewrite it as described in the next result:

Proposition 3.6. *Let F_n and ψ_n be the functions defined above. Then $Z_n(x,t)$ can be written as*

$$Z_n(x,t)=\sum_{k=0}^{n} C_k(n,a)e^{ix(1-2k/n)}e^{i\Theta_{k,n}(t)}\sin\big[(t/2)(1-2k/n-a)\big],$$

where

$$\tan\Theta_{k,n}(t):=-\frac{\sin[t(1-2k/n)^2]-\sin[ta(1-2k/n)]}{\cos[t(1-2k/n)^2]-\cos[ta(1-2k/n)]}. \tag{4}$$

Proof. The result follows from the equality

$$\psi_n(x,t)-F_n(x-at)=\sum_{k=0}^{n} C_k(n,a)e^{ix(1-2k/n)}\big[e^{it(1-2k/n)^2}-e^{ita(1-2k/n)}\big],$$

and it is based on the fact that

$$e^{it(1-2k/n)^2}-e^{ita(1-2k/n)}=\rho_{k,n}(t)e^{i\Theta_{k,n}(t)},$$

where $\Theta_{k,n}(t)$ is given by (4) and

$$\rho_{k,n}^2(t)=2-2\cos\big[(t(1-2k/n)^2-at(1-2k/n)\big].$$

With some elementary computation we get the statement. □

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